Analysis on Manifolds

Tuesday 01.02.2022, 8:30 – 10:30

You have 2 hours to complete the exam. This exam consists of 5 exercises, for a total of 90 points. You get 10 points for free.
Usage of the theory and examples from the lecture notes is allowed, with the only exceptions of the results of Exercise 8.4.7. Give a precise reference to the theory and/or exercises you use for solving the problems.

Exercise 1. (10 + 10 points)

a. Explain in your own words what distinguishes an embedded submanifold from an immersed submanifold. Can you provide an example to showcase the difference (note: a well justified picture is also enough)?

b. Let $U := \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right) \subset \mathbb{R}$. Consider the following functions $\mathbb{R} \to \mathbb{R}^3$:

$$f(t) := (\cos(t), \sin(2t), 0), \quad g(t) := (t, t^3, 0) \quad \text{and} \quad h(t) := (\sin(t), \cos(t), t).$$

Which of the curves $f(U)$, $g(\mathbb{R})$ and $h(\mathbb{R})$ is an immersed submanifold of $\mathbb{R}^3$? Which one is an embedded submanifold? Justify your answer.

Exercise 2. (10 + 10 points)

Let (x, y, z) denote the standard euclidean coordinate in $\mathbb{R}^3$. Let $\omega \in \Omega^2(\mathbb{R}^3)$ be defined by

$$\omega := zy^2\, dx \wedge dy$$

a. Let $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ by $\phi(t, u, v) = (2uv, u + v, t)$. Compute $\phi^*(\omega)$.

b. Let $X = \frac{\partial}{\partial y}$. Compute $\mathcal{L}_X\omega$.

Exercise 3. (10 points)

Let $X \in \mathfrak{X}(\mathbb{R}^3)$. In Example 7.5.10, we have shown that the exterior derivative, the musical isomorphisms ($\flat$ and $\sharp$) and the Hodge $\star$ isomorphism allow us to define $\nabla := \sharp\, d$, $\nabla\cdot := \star d \star \flat$ and $\nabla\times := \sharp \star d\, \flat$.

Show that $X = \nabla f$ is the gradient of a function $f \in C^\infty(\mathbb{R}^3)$ if and only if $\nabla \times X = 0$ everywhere.

Note: $\star : \Omega^k(\mathbb{R}^3) \to \Omega^{3-k}(\mathbb{R}^3)$ *on one-forms is given by* $\star dx = dy \wedge dz$, $\star dy = dz \wedge dx$ *and* $\star dz = dx \wedge dy$, *however you don't need to compute anything explicitly in order to solve this exercise. Think about the maps involved, their properties and their kernels...*

Exercise 4. (10 + 10 points)

Let M be a 3-dimensional manifold, let $X \in \mathfrak{X}(M)$ and $\omega \in \Omega^1(M)$ such that

$$\iota_X \omega = 1, \quad \mathcal{L}_X \omega = 0.$$

Suppose that $d\omega$ is a nowhere-vanishing form.

a. Show that $\omega \wedge d\omega$ is nowhere-vanishing (*Hint: Cartan's magic formula...*)

b. Is M orientable? Justify your answer.

Exercise 5. (10 + 10 points)

a. Compute $\int_{\mathbb{S}^2} \omega$ where $\omega \in \Omega^2(\mathbb{R}^3)$ is defined by $\omega = x^1 dx^2 \wedge dx^3 - x^2 dx^1 \wedge dx^3 + x^3 dx^1 \wedge dx^2$

b. Show that, as a differential form on $\mathbb{S}^2$, ω is closed but not exact.